CRAMÉR-RAO BOUND ON LIE GROUPS WITH OBSERVATIONS ON LIE GROUPS: APPLICATION TO $SE(2)$

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ABSTRACT

In this communication, we derive a new intrinsic Cramér-Rao bound for both parameters and observations lying on Lie groups. The expression is obtained by using the intrinsic properties of Lie groups. An exact expression is obtained for the case where parameters and observations are in $SE(2)$, the semi-direct Lie group of 2D rotation and 2D translation. To support the discussion, the proposed bound is numerically validated for a Lie group Gaussian model on $SE(2)$.

***Index Terms*—** Estimation on Lie groups, intrinsic Cramér-Rao bounds, Gaussian distribution on Lie groups.

1. INTRODUCTION

Performance bounds are crucial in various signal processing areas. Indeed, given a statistical model, these bounds provide the minimum mean square error (MSE) that an estimator can expect to achieve. In many applications, the unknown parameters' vector is constrained to respect some geometrical properties, and subsequently to lie in a Riemannian manifold. That is why estimating parameters belonging to a manifold has become a relevant issue the last decades [1, 2, 3]. Typical manifolds of interest: (1) *The manifold of symmetric definite positive (SPD) matrices*. For instance, in radar target tracking [4], the covariance of the statistical model is often unknown and can be estimated on such manifold. In image processing, pixels can be modelled by a SPD matrix [5, 6]. Then, the mean of these data have to be estimated on the SPD space. (2) *The Stiefel manifold*. In the context of blind source separation, the demixing consists in estimating an orthonormal matrix lying on the Stiefel manifold [7]. (3) *Matrix Lie groups (LGs)*. These can be found in the field of robotics and navigation, to estimate a robot attitude on $SO(2)$ or $SO(3)$ [8], or directly its pose on $SE(2)$ or $SE(3)$, for simultaneous localization and mapping [9, 10]. To evaluate the performance of an estimation problem on a manifold, it is fundamental to first define an intrinsic MSE taking into account the properties of the latter. Then, it is highly useful to build intrinsic lower bounds allowing to assess the minimum intrinsic MSE achievable. It is worth noticing that intrinsic Cramér-Rao bounds (ICRBs) have already been derived in the setting of the Riemannian manifold of SPD matrices [11, 12]. In this work, we focus on the development of intrinsic bounds for parameters belonging to matrix LGs, with observations on LGs. In [13], a non-Bayesian ICRB on LGs is proposed, which considers Euclidean observations with closed-form expression for $SO(3)$ in the context of the Wabha's problem [14]. In [15, 16], a more generic ICRB is established but only valid for symmetric or isotropic probability density functions (pdfs). Also, in a Bayesian framework, an intrinsic posterior bound on unimodular matrix LGs has been proposed in [17].

In this paper, we deal with deterministic parameters. Notice that previously derived non-Bayesian bounds in this topic are too restrictive. Indeed, in some applications, for instance in computer vision, observations are often constrained on LGs [18, 19]. In addition, existing bounds admit tractable expressions only for $SO(3)$, but other LGs are of interest, especially $SE(2)$ (the semi-direct group of translation and rotations in 2D space) [20, 21, 22], which needs to be studied. To achieve that, we propose the following generalization: we start by designing a non-Bayesian Cramér-Rao bound, which contrary to [13], applies for observations on LGs. Particularly, the demonstration is carried out by leveraging the unbiased condition for estimators on LGs. From that, we analyse and give closed-from expressions for LG $SE(2)$. Due to the properties of the latter, we gather a closed-form expression for LG Gaussian observations also lying on $SE(2)$. The proposed computations are handled in an exact way.

The communication is organized as follows: first, we remind the required background on LGs. Second, we develop the ICRB and give its explicit expression in the case of $SE(2)$. In the last section, the proposed bound is validated for the LG observation Gaussian model on $SE(2)$.

2. BACKGROUND ON LIE GROUPS

2.1. Definition

A matrix LG $G \subset \mathbb{R}^{n\times n}$ is a matrix space equipped with a structure of smooth manifold and group. Its structure of smooth manifold allows to specify the operations of integration and derivation. More precisely, we can define the notion of tangent space according to each element of G. Its structure of group allows to define an internal law which acts between each element of G. Consequently, it exists a neutral element (identity matrix) such as each element of G can be inverted. Moreover, we can leverage this internal law to connect each element of the identity tangent space to the tangent space of any element. The identity tangent space is the Lie algebra and denoted $\mathfrak{g}$. Each element of the LG and the Lie algebra are connected between them, through the logarithm and exponential applications defined, respectively, by $\mathrm{Exp}_G : \mathfrak{g} \to G$ and $\mathrm{Log}_G : G \to \mathfrak{g}$, as illustrated in Figure 1. As $\mathfrak{g}$ is isomorph to $\mathbb{R}^m$, we can define two bijections $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$. In this way, we can denote the exponential and logarithm applications such as: $\forall\, \mathbf{a} \in \mathbb{R}^m$, $\mathrm{Exp}_G^\wedge(\mathbf{a}) = \mathrm{Exp}\left([\mathbf{a}]_G^\wedge\right)$ and $\forall\, \mathbf{M} \in G$, $[\mathrm{Log}_G(\mathbf{M})]_G^\vee = \mathrm{Log}_G^\vee(\mathbf{M})$.

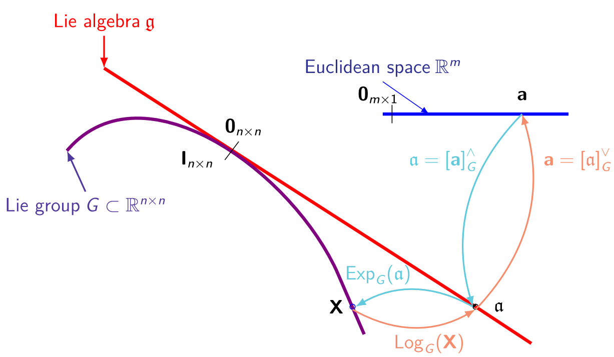

Fig. 1. Relation between $\mathbb{R}^m$, G and $\mathfrak{g}$

In order to establish the proposed bound, the Baker-Campbell-Hausdorff (BCH) formula is needed. This is a relation expressing the general non-commutativity of a LG. $\forall \mathbf{c}, \mathbf{d} \in (\mathbb{R}^m)^2$, we have that,

$$\mathrm{Log}_G^\vee(\mathrm{Exp}_G^\wedge(\mathbf{b})\,\mathrm{Exp}_G^\wedge(\mathbf{c})) = \mathbf{b} + \boldsymbol{\psi}_G(\mathbf{c})\,\mathbf{b} + O(||\mathbf{c}||^2), \tag{1}$$

where

$$\boldsymbol{\psi}_G(\mathbf{c}) = \sum_{n=0}^{+\infty} \frac{B_n}{n!}\, ad_G(\mathbf{c})^n, \tag{2}$$

denotes the inverse of the left Jacobian matrix of G, the coefficients $\{B_n\}_{n=0}^{\infty}$ are the Bernoulli numbers [23] and $ad_G(\mathbf{d})$ is the adjoint representation on $\mathfrak{g}$ of $\mathbf{d}$.

$$ad_G(\mathbf{c})\mathbf{d} = [\mathbf{c}, \mathbf{d}]\ \forall\, \mathbf{d} \in \mathbb{R}^m \tag{3}$$

where $[.,.]$ denotes the Lie bracket [24]. In the following, we assume that G is a LG with Lie algebra isomorph to $\mathbb{R}^m$.

2.2. Statistics on Lie groups

In an Euclidean context, an estimator $\widehat{\mathbf{m}}$ of the unknown parameter $\mathbf{m} \in \mathbb{R}^p$, obtained from the likelihood $p(\mathbf{z}|\mathbf{m})$, is specified by three appropriated statistical indicators: its mean $\mathbf{m}_{\widehat{\mathbf{m}}}$ such as $\int_{\mathbb{R}^s}(\widehat{\mathbf{m}} - \mathbf{m}_{\widehat{\mathbf{m}}})p(\mathbf{z}|\mathbf{m})d\mathbf{z} = \mathbf{0}$, its bias $\int_{\mathbb{R}^s}(\mathbf{m} - \widehat{\mathbf{m}})p(\mathbf{z}|\mathbf{m})\,d\mathbf{z}$ and its estimation error covariance $\int_{\mathbb{R}^s}(\mathbf{m} - \widehat{\mathbf{m}})\,(\mathbf{m} - \widehat{\mathbf{m}})^\top\, p(\mathbf{z}|\mathbf{m})d\,\mathbf{z}$. Now, consider a random observation $\mathbf{Z}$, belonging to a LG G', depending of an unknown parameter $\mathbf{M} \in G$ and generated by the likelihood $p(\mathbf{Z}|\mathbf{M})$. On LGs, the gap between two LG points $\mathbf{A}$ and $\mathbf{B}$ can be quantified by using the metric

$$\boldsymbol{l}_G{}_{\mathbf{A}}^{\mathbf{B}} = \mathrm{Log}_G^\vee\left(\mathbf{A}^{-1}\,\mathbf{B}\right) \tag{4}$$

and allows to measure the path between $\mathbf{M}$ and some estimator $\widehat{\mathbf{M}}$.

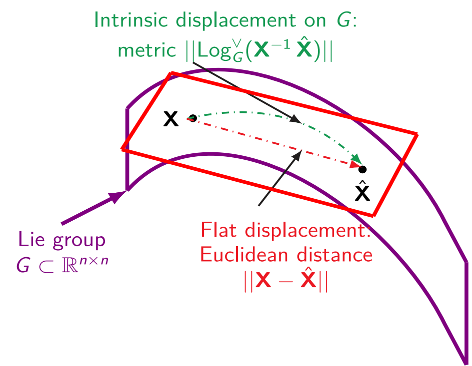

Fig. 2. Illustration of the intrinsic gap between $\mathbf{M}$ and $\widehat{\mathbf{M}}$, which takes into account the curvature of the group.

Let $\lambda_G(.)$ be Haar measure. The mean $\mathbf{M}_{\widehat{\mathbf{M}}} \in G$ is defined such that[1]:

$$\int_{G'} \boldsymbol{l}_G{}_{\mathbf{M}_{\widehat{\mathbf{M}}}}^{\widehat{\mathbf{M}}}\, p(\mathbf{Z}|\mathbf{M})\lambda_{G'}(d\mathbf{Z}) = \mathbf{0}, \tag{5}$$

its intrinsic bias $\mathbf{b}_{\mathbf{Z}|\mathbf{M}} \in \mathbb{R}^m$ given by [25]:

$$\mathbf{b}_{\mathbf{Z}|\mathbf{M}}(\mathbf{M}, \widehat{\mathbf{M}}) = \int_{G'} \boldsymbol{l}_G{}_{\widehat{\mathbf{M}}}^{\mathbf{M}}\, p(\mathbf{Z}|\mathbf{M})\lambda_{G'}(d\mathbf{Z}), \tag{6}$$

and its intrinsic estimation error covariance $\mathbf{C}_{\mathbf{Z}|\mathbf{M}} \in \mathbb{R}^{m\times m}$ defined by [15]:

$$\mathbf{C}_{\mathbf{Z}|\mathbf{M}}(\mathbf{M}, \widehat{\mathbf{M}}) = \int_{G'} \boldsymbol{l}_G{}_{\widehat{\mathbf{M}}}^{\mathbf{M}}\ \boldsymbol{l}_G{}_{\widehat{\mathbf{M}}}^{\mathbf{M}}{}^\top\ p(\mathbf{Z}|\mathbf{M})\,\lambda_{G'}(d\,\mathbf{Z}). \tag{7}$$

In the next section, we use the introduced tools on LGs in order to develop the proposed intrinsic Cramér-Rao bound.

3. INTRINSIC CRAMÉR-RAO BOUND

3.1. General expression

Let us assume a set of observations $\mathbf{Z} \in G'$ depending of an unknown parameter $\mathbf{M} \in G$ through the likelihood $p(\mathbf{Z}|\mathbf{M})$. Furthermore, let us consider that $\widehat{\mathbf{M}}$ is an unbiased estimator of $\mathbf{M}$ in the sense that,

$$\mathbf{b}_{\mathbf{Z}|\mathbf{M}}\left(\mathbf{M}, \widehat{\mathbf{M}}\right) = \mathbf{0}. \tag{8}$$

[1] For a random matrix LGs, $\mathbb{E}(f(\mathbf{X})) = \int p(\mathbf{X})\, f(X)\lambda_G(d\mathbf{X})$ where $\lambda_G(.)$ is a Haar measure

The covariance intrinsic error estimation verifies,

$$\boxed{\mathbf{C}_{\mathbf{Z}|\mathbf{M}}(\mathbf{M}, \widehat{\mathbf{M}}) \succeq \mathbf{P}} \tag{9}$$

with,

$$\boxed{\mathbf{P} = \mathbb{E}\left(\psi_G\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\right)\right) \mathcal{I}_G^{-1} \mathbb{E}\left(\psi_G\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\right)^\top\right)} \tag{10}$$

and,

$$\mathcal{I}_G = \mathbb{E}\left(\left.\frac{\partial \mathrm{l}p(\mathbf{Z}|\mathbf{M}, \boldsymbol{\delta}_1))}{\partial \boldsymbol{\delta}_1} \frac{\partial \mathrm{l}p(\mathbf{Z}|\mathbf{M}, \boldsymbol{\delta}_2))}{\partial \boldsymbol{\delta}_2}^\top \right|_{\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 = \mathbf{0}}\right), \tag{11}$$

with $\mathrm{l}p(\mathbf{Z}|\mathbf{M}, \boldsymbol{\delta}) = \log p\left(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\right)$. In the case where G is unimodular, it is possible to take advantage of an specific integration by parts on LGs [15]. By applying it, I_G can be recast:

$$-\mathbb{E}\left(\left.\frac{\partial^2 \log p\left(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\right)}{\partial \boldsymbol{\delta}_1 \partial \boldsymbol{\delta}_2}\right|_{\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 = \mathbf{0}}\right). \tag{12}$$

Proof of (10) :

In the same way that the Euclidean CRB demonstration, we define the following vector,

$$\mathbf{s} = \left[\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}{}^\top, \quad \nabla_{\mathbf{M}} L_p^\top\right]^\top. \tag{13}$$

with

$$\nabla_{\mathbf{M}} L_p = \left.\frac{\partial \log p\left(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\right)}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}. \tag{14}$$

The correlation matrix $\mathbf{R} = \mathbb{E}\left(\mathbf{s}\,\mathbf{s}^\top\right)$ can be decomposed as,

$$\mathbf{R} = \begin{bmatrix} \mathbb{E}\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\ \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}{}^\top\right) & \mathbb{E}\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\ \nabla_{\mathbf{M}} L_p^\top\right) \\ \mathbb{E}\left(\nabla_{\mathbf{M}} L_p\ \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}{}^\top\right) & \mathbb{E}\left(\nabla_{\mathbf{M}} L_p\, \nabla_{\mathbf{M}} L_p^\top\right) \end{bmatrix}, \tag{15}$$

$\mathbf{R}$ being positive semi-definite, its Schur complement satisfies this property as well. It ensues:

$$\boxed{\mathbb{E}\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\ \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}{}^\top\right) \succeq \mathbf{L}} \tag{16}$$

with

$$\mathbf{L} = \mathbb{E}\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\ \nabla_{\mathbf{M}} L_p^\top\right) \mathbb{E}\left(\nabla_{\mathbf{M}} L_p\, \nabla_{\mathbf{M}} L_p^\top\right)^{-1} \mathbb{E}\left(\nabla_{\mathbf{M}} L_p\ \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}{}^\top\right). \tag{17}$$

Expression of $\mathbb{E}\left(\boldsymbol{l}_{\mathrm{G}}{}_{\widehat{\mathbf{M}}}\ \nabla_{\mathbf{M}} L_p^\top\right)$:

By leveraging the unbiased condition (8), we obtain:

$$\left.\left(\frac{\partial}{\partial \boldsymbol{\delta}} \int_{G'} \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}_{\widehat{\mathbf{M}}}\ p(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}))\lambda_{G'}(d\mathbf{Z})\right)\right|_{\boldsymbol{\delta}=\mathbf{0}} = \mathbf{0} \tag{18}$$

By assuming that $\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}_{\widehat{\mathbf{M}}}\ p(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}))$ is bounded by a measurable function, it is possible to invert integral and derivative operators. By using classical derivative of functions,

$$\left.\left(\int_{G'} \frac{\partial}{\partial \boldsymbol{\delta}} \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}_{\widehat{\mathbf{M}}}\ p(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}))\lambda_{G'}(d\mathbf{Z}) + \int_{G'} \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}_{\widehat{\mathbf{M}}}\ \frac{\partial}{\partial \boldsymbol{\delta}} p(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}))^\top \lambda_{G'}(d\mathbf{Z})\right)\right|_{\boldsymbol{\delta}=\mathbf{0}} = \mathbf{0} \tag{19}$$

This expression can be simplified by using two tricks: first, the term $\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}_{\widehat{\mathbf{M}}}$ can be simplified by the BCH formula,

$$\frac{\partial}{\partial \boldsymbol{\delta}} \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}_{\widehat{\mathbf{M}}} = -\psi_G(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}). \tag{20}$$

Second, we know that,

$$\frac{\partial}{\partial \boldsymbol{\delta}} p(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})) = \frac{\partial}{\partial \boldsymbol{\delta}} \mathrm{l}p(\mathbf{Z}, \mathbf{M}, \boldsymbol{\delta})\, p(\mathbf{Z}|\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})), \tag{21}$$

and using (19),

$$\int_{G'} \psi_G(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}_{\widehat{\mathbf{M}}})p(\mathbf{Z}|\mathbf{M})\lambda_{G'}(d\mathbf{Z}) + \int_{G'} \boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}} \frac{\partial}{\partial \boldsymbol{\delta}} \mathrm{l}p(\mathbf{Z}|\mathbf{M}, \boldsymbol{\delta})|_{\boldsymbol{\delta}=\mathbf{0}}^\top\, p(\mathbf{Z}|\mathbf{M})\,\lambda_{G'}(d\mathbf{Z}) = \mathbf{0}. \tag{22}$$

By isolating the right-hand integral, we obtain,

$$\mathbb{E}\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\ \nabla_{\mathbf{M}} L_p^\top\right) = \mathbb{E}\left(\psi_G\left(\boldsymbol{l}_{\mathrm{G}}{}^{\mathbf{M}}_{\widehat{\mathbf{M}}}\right)\right) \tag{23}$$

By injecting the last equation in (17), we deduce the formula (10).

3.2. Application to LG-Gaussian model on $SE(2)$

In order to define statistical model with observation on LGs, we can leverage on the concentrated Gaussian distributions (CGD) on LGs, introduced in [26] and generalizing the concept of Euclidean Gaussian multivariate pdf. If $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$ and $\boldsymbol{\mu}$ a matrix defined on a LG G, then $\mathbf{X} = \boldsymbol{\mu}\,\mathrm{Exp}_G^\wedge(\boldsymbol{\epsilon})$ is distributed according to a left CGD on G, with parameters $\boldsymbol{\mu} \in G$ and $\mathbf{P} \in \mathbb{R}^{m \times m}$. If $\boldsymbol{\mu}$ is close to $\mathbf{I}$,

$$p(\mathbf{X}) \simeq \frac{1}{\sqrt{(2\,\pi)^m\,|\mathbf{P}|}} \exp^{-\frac{1}{2}||\mathrm{Log}_G^\vee(\boldsymbol{\mu}^{-1}\mathbf{X})||^2_{\mathbf{P}}}. \tag{24}$$

Let us assume that some observations $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ are distributed according to a CGD on $SE(2)$ and connected to unknown parameter $\mathbf{M} \in SE(2)$:

$$\mathbf{Z}_i = \mathbf{M}\,\mathrm{Exp}_{SE(2)}^\wedge(\boldsymbol{\epsilon}_i)\ \ \boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})\ \forall i \in \{1, \ldots n\} \tag{25}$$

with $\mathbf{\Sigma} \in \mathbb{R}^{3\times3}$. Let us consider the full observation matrix $\mathbf{Z} = \{\mathbf{Z}_1, \dots \mathbf{Z}_n\}$ and $p(\mathbf{Z}|\mathbf{M})$ the associated likelihood. It is worth noticing the expression of $\mathbf{P}$ can be simplified in this case. First, $SE(2)$ is commutative, consequently, the adjoint operator is equal to $\mathbf{0}$. Thus, $\psi_{SE(2)}(.) = \mathbf{I}$ and,

$$\mathbf{C}_{\mathbf{Z}|\mathbf{M}}(\mathbf{M}, \widehat{\mathbf{M}}) \succeq \mathcal{I}_{SE(2)}^{-1}, \tag{26}$$

where $\widehat{\mathbf{M}}$ denotes some unbiased estimator of $\mathbf{M}$.

Second, $SE(2)$ is unimodular, consequently, $\mathcal{I}_{SE(2)}$ can be computed with (12). According to the CGD expression,

$$\log p \left(\mathbf{Z} | \mathbf{M} \, \mathrm{Exp}^{\wedge}_{SE(2)}(\boldsymbol{\epsilon}_1) \;\; \mathrm{Exp}^{\wedge}_{SE(2)}(\boldsymbol{\epsilon}_2) \right) = \overbrace{K}^{\in\mathbb{R}}$$
$$- \frac{1}{2} \sum_{i=1}^{n} \left\| \boldsymbol{l}_{\mathrm{SE(2)}} \frac{\mathbf{Z}_i}{\mathbf{M}\, \mathrm{Exp}^{\wedge}_{SE(2)}(\boldsymbol{\epsilon}_1) \;\; \mathrm{Exp}^{\wedge}_{SE(2)}(\boldsymbol{\epsilon}_2)} \right\|_{\mathbf{\Sigma}} \tag{27}$$

Furthermore, the BCH formula provides that,

$$\boldsymbol{l}_{SE(2)}(\mathbf{M}\, \mathrm{Exp}^{\wedge}_{SE(2)}(\boldsymbol{\epsilon}_1) \;\; \mathrm{Exp}^{\wedge}_{SE(2)}(\boldsymbol{\epsilon}_2) \;\; , \mathbf{Z}_i) = \boldsymbol{l}_{\mathrm{SE(2)}} \frac{\mathbf{Z}_i}{\mathbf{M}}$$
$$- \underbrace{\boldsymbol{\psi}_{SE(2)}(\boldsymbol{l}_{\mathrm{SE(2)}} \frac{\mathbf{Z}_i}{\mathbf{M}})}_{=\mathbf{I}} (\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2) + O\left(||\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2||^2 \right). \tag{28}$$

By differentiating the previous expression according to $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ and taking their values to zero, we gather $\mathcal{I}_{SE(2)} = n\,\mathbf{\Sigma}^{-1}$ and $\mathbf{P} = n^{-1}\,\mathbf{\Sigma}$.

4. SIMULATION RESULTS

In sequel the proposed intrinsic bound is validated in the case of the LG-Gaussian observation model (25) for $SE(2)$, by comparing it to the intrinsic MSE (IMSE). To simulate the observations, we assume that $\mathbf{\Sigma}$ is diagonal and equal to diag $\left[\sigma_\theta^2, \sigma_x^2, \sigma_y^2\right]$. The IMSE expression is given by the trace of (7). As the latter is not tractable, it can be approximated,

$$\frac{1}{N_{mc}} \sum_{t=1}^{N_{mc}} \left\| \boldsymbol{l}_{\mathrm{SE(2)}} \frac{\widehat{\mathbf{X}}_0^t}{\mathbf{X}_0} \right\|^2 , \tag{29}$$

where N_{mc} is the number of realizations and $\widehat{\mathbf{X}}_0^t$ the t^{th} realization of the estimator. The latter is gathered by determining the likelihood maximum of $p(\mathbf{Z}|\mathbf{M})$. It amounts to find the minima of the criterion:

$$-2 \log p(\mathbf{Z}|\mathbf{M}) = \sum_{i=1}^{n} \left\| \boldsymbol{l}_{\mathrm{SE(2)}} \frac{\mathbf{Z}_i}{\mathbf{M}} \right\|_{\mathbf{\Sigma}}^2 \tag{30}$$

. To obtain a sufficiently accurate estimator, a Gauss-Newton algorithm on LGs is performed [27]. Concerning the ICRB, it is obtained by computing the trace of $\mathbf{P}$. To validate the consistence of the bound, we first compare the IMSE and the ICRB for different values of translation components of observation noise. To succeed that, we assume that $\sigma_x = \sigma_y = \sigma_d$. In Figure 3, we observe that the proposed ICRB is consistent because it minores the IMSE whatever the value of σ_d. On the other hand, we remark that for high variance noise, the difference between the IMSE and the ICRB increases. Thus, it numerically proves that the estimation algorithm can not draw an optimal estimator in very noisy conditions. In Figure 4, we draw the same variables for different number of observations by staring the covariance $\mathbf{\Sigma}$. As previously, we see that the ICRB admits a consistent behaviour. Furthermore, we remark the IMSE tends to the ICRB for large number of observations. Its interesting to note that we find the same behaviour as in the case of the Euclidean Gaussian model. This asymptotic convergence can be regarded as a theorical result due to the fact that the ICRB is computed in a exact way.

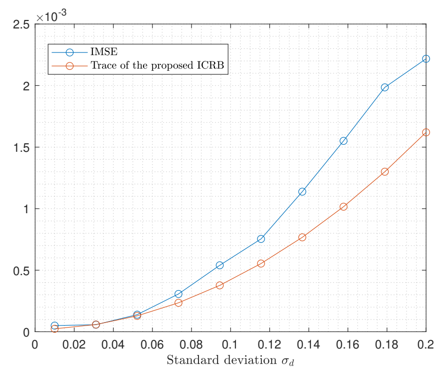

Fig. 3. Evolution of the IMSE and ICRB as a function of σ_d, with $\sigma_\theta = 10^{-3}$ rad, $n = 50$ and $N_{mc} = 500$.

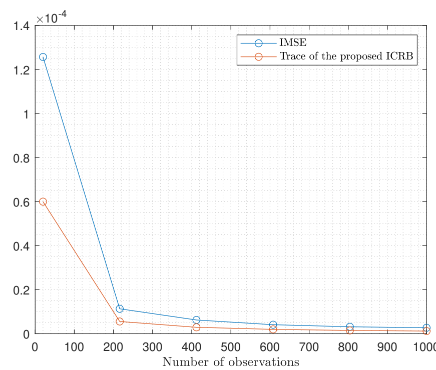

Fig. 4. Evolution of the IMSE and ICRB as a function of n with $\sigma_\theta = 10^{-3}$ rad, $\sigma_d = 10^{-2}$ m and $N_{mc} = 500$.

5. CONCLUSIONS

In this communication, we derived an intrinsic Cramér-Rao bound which applies for observations belonging to LGs, which generalizes known results in the literature. In addition, we obtained an exact expression in the case of the Gaussian model on LGs for observations and parameter lying on $SE(2)$. The consistency of the proposed bound was validated by numerical simulations. A perspective of this work would be to derive closed-form expressions for other LGs of interests. More precisely, when the covariance of the model is unknown, it would be relevant to derive bounds on the latter taking into account its LG structure.

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